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# BESOV 正則性を持つ多重線形フーリエマルチプライヤーについて

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In this note, we consider weighted norm inequalities for multilinear Fourier multipliers with critical Besov regularity. This is a joint work with Naohito Tomita (Osaka University).

## 1. Introduction

For $m \in L^\infty(\mathbb{R}^{Nn})$, the $N$-linear Fourier multiplier operator $T_m$ is defined by

$$T_m(f_1, \dots, f_N)(x) = \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} e^{ix\cdot(\xi_1+\dots+\xi_N)} m(\xi) \widehat{f_1}(\xi_1) \dots \widehat{f_N}(\xi_N)\, d\xi$$

for $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$, where $x \in \mathbb{R}^n$, $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$ and $d\xi = d\xi_1 \dots d\xi_N$. Let $\Psi \in \mathcal{S}(\mathbb{R}^{Nn})$ be such that

$$\text{(1)} \qquad \operatorname{supp}\Psi \subset \left\{\xi \in \mathbb{R}^{Nn} : 1/2 \le |\xi| \le 2\right\}, \quad \sum_{k\in\mathbb{Z}} \Psi(\xi/2^k) = 1, \quad \xi \in \mathbb{R}^{Nn} \setminus \{0\},$$

and set

$$m_j(\xi_1, \dots, \xi_N) = m(2^j\xi_1, \dots, 2^j\xi_N)\Psi(\xi_1, \dots, \xi_N), \quad j \in \mathbb{Z}.$$

We denote by $\|T_m\|_{L^{p_1}(w_1)\times\dots\times L^{p_N}(w_N)\to L^p(w)}$ the smallest constant $C$ satisfying

$$\|T_m(f_1, \dots, f_N)\|_{L^p(w)} \le C \prod_{i=1}^{N} \|f_i\|_{L^{p_i}(w_i)}, \quad f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$$

(see Section 2 for the definition of function spaces).

In the unweighted case, Tomita [4] proved a Hörmander type multiplier theorem for multilinear operators, namely, if $s > Nn/2$ then

$$\|T_m\|_{L^{p_1}(\mathbb{R}^n)\times\dots\times L^{p_N}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^s(\mathbb{R}^{Nn})}$$

for $1 < p_1, \dots, p_N, p < \infty$ satisfying $1/p_1 + \dots + 1/p_N = 1/p$, where $H^s(\mathbb{R}^{Nn})$ is the Sobolev space of usual type. Grafakos-Si [3] extended this result to the case $p \le 1$ by using the $L^r$-based Sobolev spaces, $1 < r \le 2$. Let $1 < p_1, \dots, p_N < \infty$ and $1/p_1 + \dots + 1/p_N = 1/p$. In the weighted case, Fujita-Tomita [2] proved that if $n/2 < s_j \le n$, $p_j > n/s_j$ and $w_j \in A_{p_j s_j/n}$, $1 \le j \le N$, then

$$\text{(2)} \qquad \|T_m\|_{L^{p_1}(w_1)\times\dots\times L^{p_N}(w_N)\to L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)},$$

where $w = w_1^{p/p_1} \dots w_N^{p/p_N}$ and $H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)$ is the Sobolev space of product type.

The following is our main result which can be understood as a limiting case $s_j = n/2$, $1 \le j \le N$ in (2).

**Theorem 1.1.** *Let $2 < p_1, \dots, p_N < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. Assume $w_j \in A_{p_j/2}$, $j = 1, \dots, N$, and set $w = w_1^{p/p_1} \dots w_N^{p/p_N}$. Then*

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{B^{(n/2,\dots,n/2)}_{2,1}((\mathbb{R}^n)^N)}.$$

It should be remarked that

$$H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N) \hookrightarrow B^{(n/2,\dots,n/2)}_{2,1}((\mathbb{R}^n)^N) \hookrightarrow L^\infty(\mathbb{R}^{Nn}), \quad s_1, \dots, s_N > n/2.$$

## 2. Function spaces

Let $0 < p < \infty$ and $w \ge 0$. The weighted Lebesgue space $L^p(w)$ consists of all measurable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx\right)^{1/p} < \infty.$$

Let $1 < p < \infty$. We say that a weight $w$ belongs to the Muckenhoupt class $A_p$ if

$$\sup_B \left(\frac{1}{|B|}\int_B w(x)\, dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}\, dx\right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^n$, $|B|$ is the Lebesgue measure of $B$ and $p'$ is the conjugate exponent of $p$, that is, $1/p + 1/p' = 1$. It is well known that the Hardy-Littlewood maximal operator $M$ is bounded on $L^p(w)$ if and only if $w \in A_p$([1, Theorem 7.3]).

For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left|(1+|\xi|^2)^{s/2}\widehat{f}(\xi)\right|^2 d\xi\right)^{1/2} < \infty.$$

The norm of the Sobolev space of product type $H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)$, $s_1, \dots, s_N \in \mathbb{R}$, for $F \in \mathcal{S}'((\mathbb{R}^n)^N)$ is also defined by

$$\|F\|_{H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)} = \left(\int_{(\mathbb{R}^n)^N} \left|(1+|\xi_1|^2)^{s_1/2}\dots(1+|\xi_N|^2)^{s_N/2}\widehat{F}(\xi)\right|^2 d\xi\right)^{1/2},$$

where $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$.

We recall the definition of Besov spaces of usual and product types, respectively. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be as in (1) with $d = n$, and set $\psi_0(\eta) = 1 - \sum_{k=1}^\infty \psi(\eta/2^k)$, $\psi_k(\eta) = \psi(\eta/2^k)$, $k \ge 1$. Note that $\operatorname{supp}\psi_0 \subset \{\eta \in \mathbb{R}^n : |\eta| \le 2\}$, $\operatorname{supp}\psi_k \subset \{\eta \in \mathbb{R}^n : 2^{k-1} \le |\eta| \le 2^{k+1}\}$, $k \ge 1$, and $\sum_{k=0}^\infty \psi_k(\eta) = 1$, $\eta \in \mathbb{R}^n$. For $1 \le p, q \le \infty$ and $s \in \mathbb{R}$, the Besov space $B^s_{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{k=0}^\infty 2^{ksq}\|\psi_k(D)f\|_{L^p}^q\right)^{1/q} < \infty,$$

where $\psi_k(D)f = \mathcal{F}^{-1}[\psi_k \widehat{f}]$. The norm of the Besov space of product type $B_{p,q}^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)$, $s_1,\dots,s_N \in \mathbb{R}$, for $F \in \mathcal{S}'((\mathbb{R}^n)^N)$ is also defined by

$$\|F\|_{B_{p,q}^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)} = \left( \sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1 s_1+\cdots+k_N s_N)q} \left\| \Psi_{(k_1,\dots,k_N)}(D)F \right\|_{L^p}^q \right)^{1/q},$$

where

$$\begin{aligned}\Psi_{(k_1,\dots,k_N)}(\xi) &= (\psi_{k_1} \otimes \cdots \otimes \psi_{k_N})(\xi) \\ &= \psi_{k_1}(\xi_1) \times \cdots \times \psi_{k_N}(\xi_N), \quad \xi = (\xi_1, \dots \xi_N) \in (\mathbb{R}^n)^N.\end{aligned}$$

## 3. Key estimate

The following lemma is a key estimate in the proof of Theorem 1.1.

**Lemma 3.1.** *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\phi(x) = \phi(-x)$, $x \in \mathbb{R}^n$, and $\phi(x) = 1$ on $\{x \in \mathbb{R}^n : |x| \le 2\}$. Then*

$$\begin{aligned}&|T_{m(\cdot/2^j)}(f_1,\dots,f_N)(x)| \\ &\lesssim \sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \|\Psi_{(k_1,\dots,k_N)}(D)m\|_{L^2} \prod_{i=1}^{N} \left\{ (|\phi|^2)_{(k_i-j)} * |f_i|^2(x) \right\}^{1/2}\end{aligned}$$

*for $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, where $(|\phi|^2)_{(j)}(x) = 2^{-jn}|\phi(2^{-j}x)|^2$.*

## References


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